Location of Incenters and Fermat Points in Variable Triangles

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Introduction In 1765, Euler proved that several important centers of a triangle are collinear; the line containing these points is named after him. The incenter I of a triangle, however, is generally not on this line. Less than twenty years ago, Andrew P. Guinand discovered that although I is not necessarily on the Euler line, it is always fairly close to it. Guinand's theorem [**5**] states that for any non-equilateral triangle, the incenter lies inside, and the excenters lie outside, the circle whose diameter joins the centroid to the orthocenter; henceforth the orthocentroidal circle. Furthermore, Guinand constructed a family of locus curves for I which cover the interior of this circle twice, showing that there are no other restrictions on the positions of the incenter with respect to the Euler line.

Here we show that the Fermat point also lies inside the orthocentroidal circle; this suggests that the neighborhood of the Euler line may harbor more secrets than was previously known. We also construct a simpler family of curves for I, covering the interior once only except for the nine-point center N, which corresponds to the limiting case of an equilateral triangle.

Triangle geometry is often believed to be exhausted, although both Davis [**3**] and Oldknow [**6**] have expressed the hope that the use of computers may revive it. New results do appear occasionally, such as Eppstein's recent construction of two new triangle centers [**4**]. This article establishes some relations among special points of the triangle, which were indeed found by using computer software.

The locus of the incenter To any plane triangle ABC there are associated several special points, called *centers*. A few of these, in the standard notation [**1**], are: the centroid G, where the medians intersect; the orthocenter H, where the altitudes meet; the centers of the inscribed and circumscribed circles, called the incenter I and the circumcenter O; and the nine-point center N, half-way between O and H. The radii of the circumcircle and the incircle are called R and r, respectively.

If equilateral triangles BPC, AQC and ARB are constructed externally on the sides of the triangle ABC, then the lines AP, BQ and CR are concurrent and meet at the Fermat point T. This point minimizes the distance $TA+TB+TC$ for triangles whose largest angle is $\leq 120°$ [**2**].

The points O, G, N, and H lie (in that order) on a line called the Euler line, and $OG : GN : NH = 2 : 1 : 3$. They are distinct unless ABC is equilateral. The circle whose diameter is GH is called the *orthocentroidal circle*.

Guinand noticed that Euler's relation $OI^2 = R(R-2r)$ [**1**, p. 85] and Feuerbach's theorem $IN = \frac{1}{2}R - r$ [**1**, p. 105] together imply that

$$OI^2 - 4\,IN^2 = R(R-2r) - (R-2r)^2 = 2r(R-2r) = \frac{2r}{R}\,OI^2 > 0.$$

Therefore, $OI > 2\,IN$. The locus of points P for which $OP = 2\,PN$ is a circle of Apollonius; since $OG = 2\,GN$ and $OH = 2\,HN$, this is the orthocentroidal circle. The inequality $OI > 2\,IN$ shows that I lies in the interior of the circle [**5**].

Guinand also showed that the angle cosines $\cos A$, $\cos B$, $\cos C$ of the triangle satisfy the following cubic equation:

$$\rho^4(1-2x)^3 + 8\rho^2\sigma^2 x(3-2x) - 16\sigma^4 x - 4\sigma^2\kappa^2(1-x) = 0, \tag{1}$$

where $OI = \rho$, $IN = \sigma$ and $OH = \kappa$. We exploit this relationship below.

The relation $OI > 2\,IN$ can be observed on a computer with the software *The Geometer's Sketchpad*®, that allows tracking of relative positions of objects as *one* of them is moved around the screen. Let us fix the Euler line by using a Cartesian coordinate system with O at the origin and H at $(3,0)$. Consequently, $G = (1,0)$ and $N = (1.5,0)$. To construct a triangle with this Euler line, we first describe the circumcircle $\odot(O,R)$, centered at O with radius $R > 1$ —in order that G lie in the interior— and choose a point A on this circle. If AA' is the median passing through A, we can determine A' from the relation $AG : GA' = 2 : 1$; then BC is the chord of the circumcircle that is bisected perpendicularly by the ray OA'.

It is not always possible to construct ABC given a fixed Euler line and a circumradius R. If $1 < R < 3$ then there is an arc on $\odot(O,R)$ on which an arbitrary point A yields an A' outside the circumcircle, which is absurd. If U and V are the intersections of $\odot(O,R)$ with the orthocentroidal circle, and if UY and VZ are the chords of $\odot(O,R)$ passing through G, then A cannot lie on the arc ZY of $\odot(O,R)$, opposite the orthocentroidal circle (Figure 1). OYG and NUG are similar triangles. Indeed, $YO = UO = 2\,UN$ and $OG = 2\,GN$, so $YG : GU = 2 : 1$; in the same way,

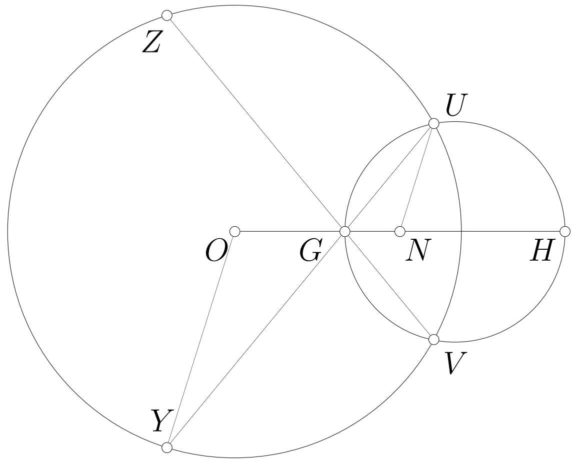


Figure 1: The forbidden arc ZY

$ZG : GV = 2 : 1$. Hence, if $A = Y$ or $A = Z$, then $A' = U$ or $A' = V$ respectively, and ABC degenerates into the chord UY or VZ. If A lies on the arc ZY, opposite the orthocentroidal circle, A' will be outside $\odot(O, R)$.

Another formula of Guinand [**5**],

$$OH^2 = R^2(1 - 8\cos A \cos B \cos C), \tag{2}$$

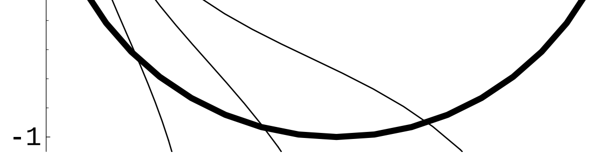
shows that the forbidden arc appears if and only if $\cos A \cos B \cos C < 0$, that is, if and only if $1 < R < 3 = OH$. The triangle ABC is obtuse-angled in this case.

Once A is chosen and the triangle constructed, we can find I by drawing the angle bisectors of ABC and marking their intersection. *The Geometer's Sketchpad* will do this automatically, and we can also ask it to draw the locus which I traces as we move the point A around the circumcircle. The idea of parameterizing the loci with R, instead of an angle of the triangle (as Guinand did) was inspired by the drawing tools available in *The Geometer's Sketchpad*. This locus turns out to be a quartic curve, as follows.

Proposition 1. *In the coordinate system described above, the incenter of ABC is on the curve*

$$(x^2 + y^2)^2 = R^2\,[(2x - 3)^2 + 4y^2]. \tag{3}$$

Proof. From equation (2) we see that, once we fix R, the product $\cos A \cos B \cos C$ is fixed. Using Viète's formulas, this product is obtained from the constant term of



Guinand's equation (1):

$$\cos A \cos B \cos C = \frac{1}{8}\left(1 - \frac{4\sigma^2\kappa^2}{\rho^4}\right),$$

so that

$$\rho^4(1 - 8\cos A \cos B \cos C) = 4\sigma^2\, OH^2. \tag{4}$$

Now, I is a point of intersection of the two circles

$$\rho^2 = x^2 + y^2 \quad \text{and} \quad \sigma^2 = (x - \tfrac{3}{2})^2 + y^2.$$

We get (3) by substituting these and (2) in (4), and dividing through by the common factor $1 - 8\cos A \cos B \cos C$, which is positive since $O \neq H$ (the equilateral case is excluded). □

Figure 2: Locus curves for I with fixed R

Figure 2 is a *Mathematica* plot of the curves (3) with the orthocentroidal circle.

Every point on a locus inside the orthocentroidal circle is the incenter of a triangle. When $R > 3$, the locus is a lobe entirely inside the circle; if the point A travels around the circumcircle once, then I travels around the lobe three times, since A will pass through the three vertices of each triangle with circumradius R.

When $1 < R < 3$, the interior portion of (3) isshaped like a bell (Figure 2). Let A travel along the allowable arc from Y to Z, passing through V and U; then I travels along the bell from U to V, back from V to U, and then from U to V again. While A moves from V to U, the orientation of the triangle ABC is reversed. When $R = 3$, the locus closes at H and one vertex B or C also coincides with H; the triangle is right-angled. If A moves once around the circumcircle, starting and ending at H, I travels twice around the lobe.

Proposition 2. *The curves (3), for different values of R, do not cut each other inside the orthocentroidal circle, and they fill the interior except for the point N.*

Proof. Let (a, b) be inside the orthocentroidal circle, that is,

$$(a-2)^2 + b^2 < 1. \tag{5}$$

If (a, b) also lies on one of the curves (3), then

$$R = \sqrt{\frac{(a^2+b^2)^2}{(2a-3)^2+4b^2}}.$$

There is only one positive value of R and thus *at most* one curve of the type (3) on which (a, b) can lie. Now we show (a, b) lies on *at least* one curve of type (3); to do that, we need to show that given (5), $R > 1$. We need only prove

$$(a^2+b^2)^2 > (2a-3)^2 + 4b^2. \tag{6}$$

Indeed, $(2a-3)^2 + 4b^2 = 0$ only if $(a, b) = (\frac{3}{2}, 0)$; this point is N. It cannot lie on a locus (3), in fact it corresponds to the limiting case of an equilateral triangle as $R \to \infty$.

The inequality (5) can be restated as

$$a^2 + b^2 < 4a - 3, \tag{7}$$

and (6) as

$$\left(a^2+b^2\right)^2 - 4a^2 + 3(4a-3) - 4b^2 > 0. \tag{8}$$

From (7) it follows that

$$\begin{aligned}(a^2+b^2)^2 - 4a^2 + 3(4a-3) - 4b^2 &> (a^2+b^2)^2 - 4a^2 + 3(a^2+b^2) - 4b^2 \\ &= (a^2+b^2)(a^2+b^2-1).\end{aligned}$$

But $a^2 + b^2 > 1$ since (a, b) is inside the orthocentroidal circle; therefore, (6) is true. □

The whereabouts of the Fermat point The same set-up, a variable triangle with fixed Euler line and circumcircle, allows us to examine the loci of other triangle centers. Further experimentation with *The Geometer's Sketchpad* suggests that the Fermat point T also lies inside the orthocentroidal circle in all cases.

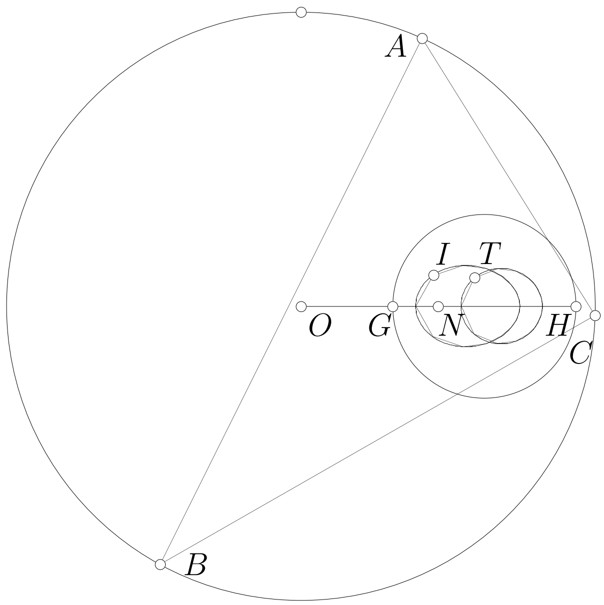


Figure 3: Locus of the Fermat point T

THEOREM 1. *The Fermat point of any non-equilateral triangle lies inside the orthocentroidal circle.*

Proof. Given a triangle ABC with the largest angle at A, we can take a coordinate system with BC as the x-axis and A on the y-axis. Then $A = (0, a)$, $B = (-b, 0)$, $C = (c, 0)$ where a, b and c are all positive. Let BPC and AQC be the equilateral triangles constructed externally over BC and AC respectively (Figure 4). Then $P = (\frac{1}{2}(c-b), -\frac{1}{2}\sqrt{3}(b+c))$ and $Q = (\frac{1}{2}(\sqrt{3}\,a + c), \frac{1}{2}(a + \sqrt{3}\,c))$.

The coordinates of T can be found by writing down the equations for the lines AP and BQ and solving them simultaneously. After a little work, we get $T = \left(\dfrac{u}{d}, \dfrac{v}{d}\right)$, where

$$\begin{aligned} u &= (\sqrt{3}\,bc - \sqrt{3}\,a^2 - ac - ab)(b - c), \\ v &= (a^2 + \sqrt{3}\,ab + \sqrt{3}\,ac + 3bc)(b + c), \\ d &= 2\sqrt{3}(a^2 + b^2 + c^2) + 6ac + 6ab + 2\sqrt{3}\,bc. \end{aligned} \tag{9}$$

The perpendicular bisectors of BC and AC intersect at the circumcenter $O =$

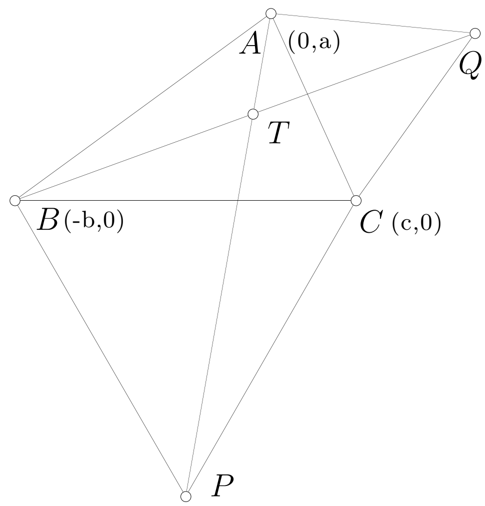


Figure 4: Finding the Fermat point

$(\frac{1}{2}(c-b), (a^2-bc)/2a)$. The nine-point center N is the circumcenter of the triangle whose vertices are the midpoints of the sides of ABC; we can deduce that $N = (\frac{1}{4}(c-b), (a^2+bc)/4a)$.

To show that T lies inside the orthocentroidal circle, we must show that $OT > 2\,NT$, or

$$OT^2 > 4\,NT^2. \tag{10}$$

In coordinates, this inequality takes the equivalent form:

$$\left[\frac{u}{d} - \left(\frac{c-b}{2}\right)\right]^2 + \left[\frac{v}{d} - \left(\frac{a^2-bc}{2a}\right)\right]^2 > 4\left[\frac{u}{d} - \left(\frac{c-b}{4}\right)\right]^2 + 4\left[\frac{v}{d} - \left(\frac{a^2+bc}{4a}\right)\right]^2,$$

or, multiplying by $(2ad)^2$,

$$[2au - ad(c-b)]^2 + [2av - d(a^2-bc)]^2 > [4au - ad(c-b)]^2 + [4av - d(a^2+bc)]^2.$$

After expanding and canceling terms, this simplifies to

$$4a^2du(c-b) + 4adv[(2a^2+2bc) - (a^2-bc)] - 4a^2d^2bc > 12a^2u^2 + 12a^2v^2,$$

or better,

$$adu(c-b) + dv(a^2+3bc) - abcd^2 - 3au^2 - 3av^2 > 0. \tag{11}$$

One way to verify this inequality is to feed the equations (9) into *Mathematica*, which expands and factors the left hand side of (11) as

$$2(b+c)(\sqrt{3}\,a^2 + \sqrt{3}\,b^2 + \sqrt{3}\,c^2 + \sqrt{3}\,bc + 3ab + 3ac)(a^4 + a^2b^2 - 8a^2bc + a^2c^2 + 9b^2c^2).$$

The first three factors are positive. The fourth factor can be expressed as the sum of two squares,

$$(a^2 - 3bc)^2 + a^2(b - c)^2,$$

and could be zero only if $a^2 = 3bc$ and $b = c$, so that $a = \sqrt{3}\,b$. This gives an equilateral triangle with side $2b$. Since the equilateral case is excluded, all the factors are positive, which shows that (11) is true, and therefore (10) holds. □

Varying the circumradius R and the position of the vertex A with *The Geometer's Sketchpad* reveals a striking parallel between the behavior of the loci of T and those of I. It appears that the loci of T also foliate the orthocentroidal disc, never cutting each other, in a similar manner to the loci of I (Figure 2). The locus of T becomes a lobe when $R = 3$, as is the case with the locus of I. Furthermore, the loci of T close in on the center of the orthocentroidal circle as $R \to \infty$, just as the loci of I close in on N (Figure 2).

It is difficult to prove these assertions with the same tools used to characterize the loci of I, because we lack an equation analogous to (1) involving T instead of I. A quick calculation for non-equilateral isosceles triangles, however, shows that T can be anywhere on the segment GH except for its midpoint. This is consistent with the observation that the loci of T close in on the center of the orthocentroidal circle.

Consider a system of coordinates like those of Figure 4. Let $b = c$ so that ABC is an isosceles triangle. In this case, by virtue of (9), $T = (0, b/\sqrt{3})$. For this choice of coordinates, $G = (0, a/3)$ and $H = (0, b^2/a)$. T lies on the Euler line, which can be parametrized by $(1-t)G + tH$ for real t. This requires that

$$(1-t)\frac{a}{3} + t\,\frac{b^2}{a} = \frac{b}{\sqrt{3}},$$

for some real t. Solving for t, this becomes

$$t = \frac{a^2 - \sqrt{3}\,ab}{a^2 - 3b^2} = \frac{a}{a + \sqrt{3}\,b},$$

unless $a = \sqrt{3}\,b$. This case is excluded since ABC is not equilateral. Note that $t \to \frac{1}{2}$ as $a \to \sqrt{3}\,b$. Thus, t takes real values between 0 and 1 except for $\frac{1}{2}$, so T can be anywhere on the segment GH except for its midpoint.

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